NESA No. | | | | | | | | | |

Class: 12MTX21______

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2021

YEAR 12

AP4

MATHEMATICS EXTENSION 1

Time allowed – 2 hours plus 10 minutes reading time

General Instructions

- Attempt all questions
- Write your name and NESA number on the question paper
- Write using black pen
- NESA approved calculators may be used
- The NESA reference sheet has been provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 6 – 10)

- Attempt Questions 11 – 14
- Each question must be written in a new booklet clearly marked with your NESA number, class and question number eg. Question 11, Question 12, Question 13 or Question 14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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1. What is the projection of $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ on to $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{j}$?

(A) $-\frac{20}{17}\underset{\sim}{i} - \frac{30}{17}\underset{\sim}{j}$

(B) $-\frac{10}{17}\underset{\sim}{i} + \frac{40}{17}\underset{\sim}{j}$

(C) $-\frac{20}{13}\underset{\sim}{i} - \frac{30}{13}\underset{\sim}{j}$

(D) $-\frac{10}{13}\underset{\sim}{i} + \frac{40}{13}\underset{\sim}{j}$

2. Which of the following is an expression for $\int \sin^2 4x dx$?

(A) $\frac{x}{2} - \frac{1}{16}\sin 8x + C$

(B) $\frac{x}{2} + \frac{1}{16}\sin 8x + C$

(C) $x - \frac{1}{8}\sin 8x + C$

(D) $x + \frac{1}{8}\sin 8x + C$

3. Which one of the following differential equations has $y = 5xe^{2x}$ as a solution?

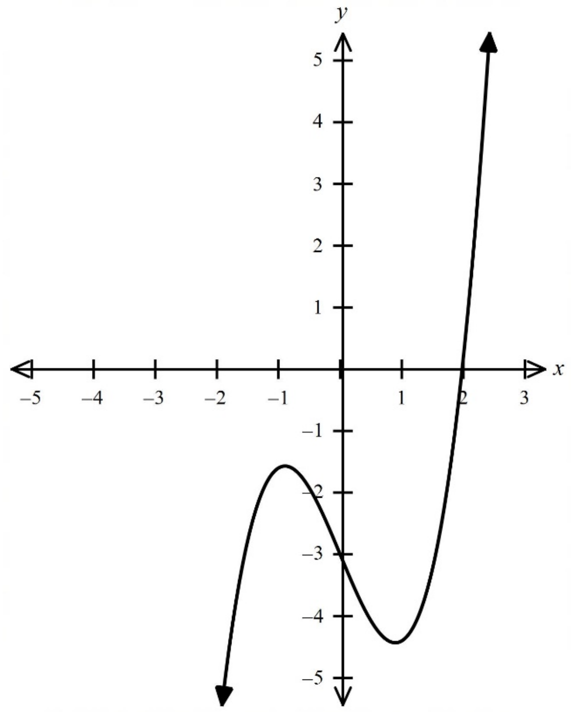
(A) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$

(B) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 0$

(C) $\frac{d^2y}{dx^2} - 4y = 20e^{2x}$

(D) $\frac{d^2y}{dx^2} - 4y = e^{2x}$

4. Consider the graph of $y = f(x)$ shown here:



Which one of the following would have 2 more roots than $f(x)$?

(A) $y = -2 \times f(x)$

(B) $y = f(x) + 3$

(C) $y = f^{-1}(x)$

(D) $y = f(x + 3)$

5. Which of the following is an expression for $\int \tan x \, dx$?

(A) $\sec^2 x + c$

(B) $\ln|\cos x| + c$

(C) $\ln|\sec x| + c$

(D) $\frac{1}{2}\tan^2 x + c$

6. If $f(x) = \frac{3+e^{2x}}{5}$ where $f(x) > \frac{5}{3}$, which of the following is $f^{-1}(x)$?

(A) $\ln(5x-3)$

(B) $\frac{1}{2}\ln(5x-3)$

(C) $\ln(5x) - \ln 3$

(D) $\frac{1}{2}\ln(5x) - \ln 3$

7. A curve is defined parametrically by $x = -\ln t$, $y = \cos 2t$ for $t > 0$.
At what approximate value of x does the curve cross the x-axis for the first time as t increases from zero?

(A) - 1.7

(B) - 1.37

(C) - 0.86

(D) 0.24

8. A company has a board of twelve directors. Six of these were selected at random to be candidates in the election for the positions of President, Vice-President and Treasurer.
In how many ways can these three senior positions be filled?

(A) ${}^{12}P_3$

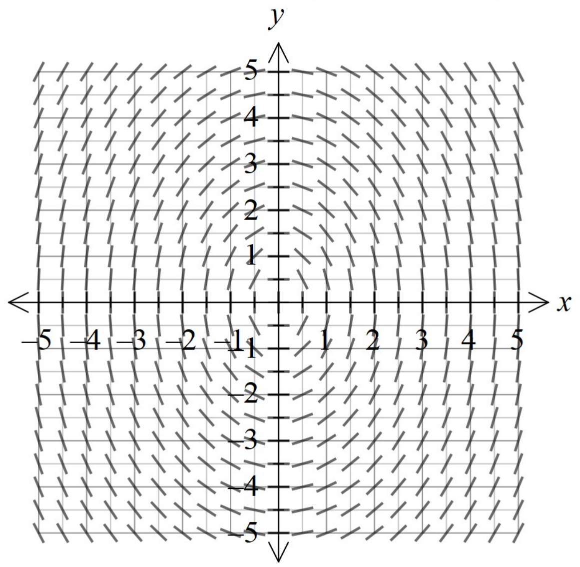
(B) $\frac{12!}{3!}$

(C) $\frac{{}^{12}P_6}{3!}$

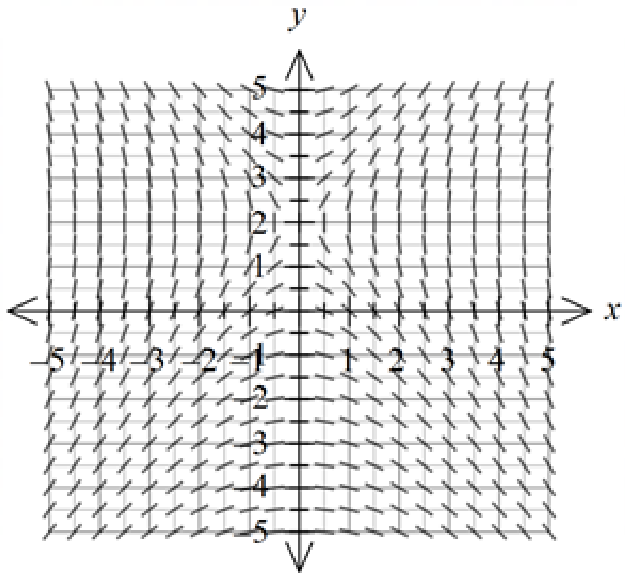
(D) $\frac{{}^{12}P_6}{3!3!}$

9. Which of the following best represents the direction field for the differential equation $\frac{dy}{dx} = \frac{-2x}{y-2}$?

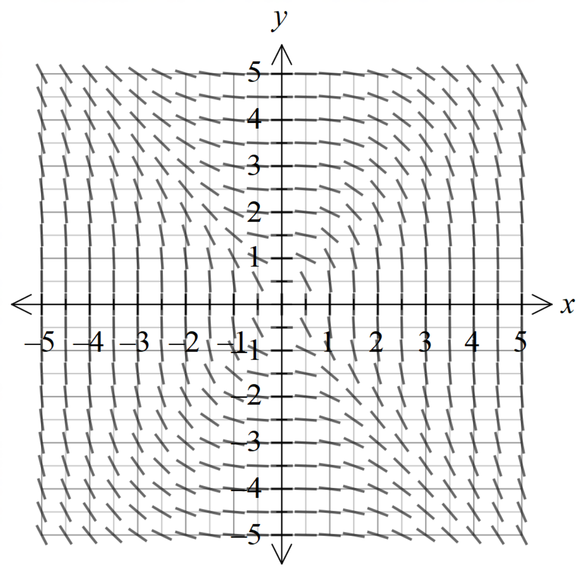
(A)



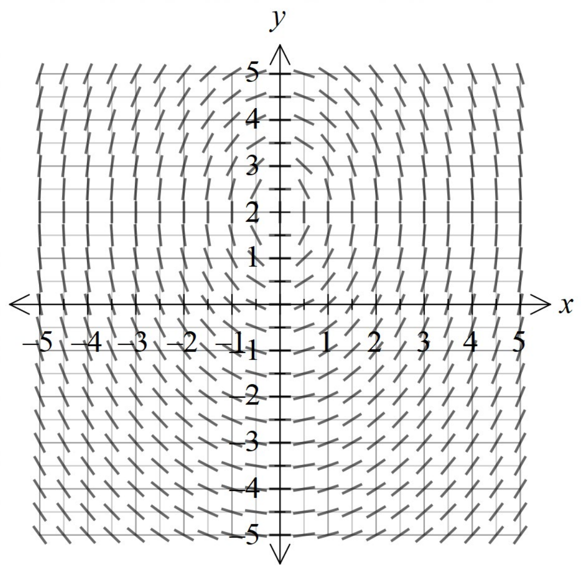
(B)



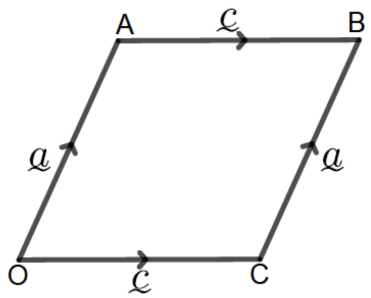
(C)



(D)



10. The diagram shows *OABC*, a rhombus in which $\overrightarrow{OA} = \overrightarrow{CB} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \overrightarrow{AB} = \underset{\sim}{c}$



To prove that the diagonals of *OABC* are perpendicular, what are you required to show?

(A) $\left(\underset{\sim}{a} + \underset{\sim}{c}\right) \cdot \left(\underset{\sim}{a} + \underset{\sim}{c}\right) = 0$

(B) $\left(\underset{\sim}{a} - \underset{\sim}{c}\right) \cdot \left(\underset{\sim}{a} - \underset{\sim}{c}\right) = 0$

(C) $\left(\underset{\sim}{a} - \underset{\sim}{c}\right) \cdot \left(\underset{\sim}{a} + \underset{\sim}{c}\right) = 0$

(D) $\underset{\sim}{a} \cdot \underset{\sim}{c} = 0$

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Instructions

- Answer each question in the appropriate booklet. Extra writing booklets are available.
- In Questions 11-14 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 – Start a new booklet (15 marks) **Marks**

(a) Solve $\dfrac{x}{x-3} \leq 4$ for x. **2**

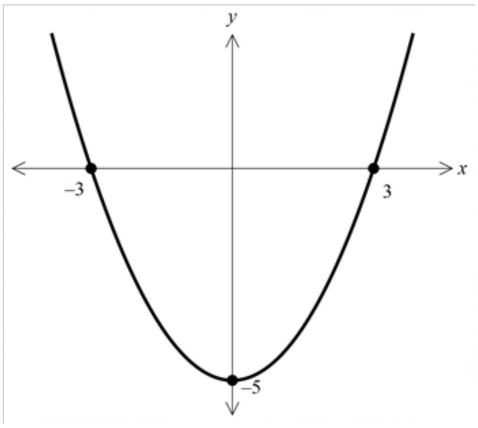
(b) α, β and γ are the roots of the polynomial $P(x) = 2x^3 - 4x^2 + x + 5$.

Calculate:

(i) $3\alpha + 3\beta + 3\gamma - 4\alpha\beta\gamma$ **2**

(ii) $\alpha^2 + \beta^2 + \gamma^2$ **2**

(c) The diagram shows the graph of $y = f(x)$.



(i) Sketch $y = f^{-1}(x)$ after making an appropriate restriction on the domain of $f(x)$. **2**

(ii) On a separate diagram, sketch $y = \dfrac{1}{|f(x)|}$. **2**

Question 11 continued on page 7

Question 11 continued

(d) Determine the values of a and b so that $x^4 - 3x^3 + ax^2 + bx - 1$ is divisible by $(x-1)^2$. **3**

(e) Find the angle between the vectors $\underset{\sim}{\mathbf{a}} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and $\underset{\sim}{\mathbf{b}} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, correct to the nearest degree. **2**

End of Question 11 – Start a new booklet

Question 12 – Start a new booklet (15 marks) **Marks**

(a) In a particular circuit, the current I amperes is given by:

$$I = 4\sin\theta - 3\cos\theta$$

Where θ is an angle related to the voltage and $\theta > 0$.

(i) Express I in the form $I = R\sin(\theta - \alpha)$ where $R > 0$ and $0^o \leq \alpha \leq \frac{\pi}{2}$. **2**

(ii) Find the value of θ for which the greatest value of I first occurs. **2**
Justify your answer mathematically.

(b) Water is poured into a container at a rate of 8 cm^3s^{-1} and the volume of the water in the container is given by: **3**

$$V = \frac{3}{2}(h^2 + 8h)$$

where h is the depth of the water in centimetres.

Find the rate of change of the depth of the water when V = 72 cm^3.

(c) Evaluate $\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \cos^2 2\theta \sin 2\theta \; d\theta$. Use the substitution $u = \cos 2\theta$. **3**

(d) Use the principle of mathematical induction to prove that for all $n \geq 1$, **3**

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$

(e) Use the pigeonhole principle to determine how many integers must be selected from the numbers 5, 6, 7, 8, 9 and 10 so that at least 2 of the numbers will have a difference of 3? **2**

End of Question 12 – Start a new booklet

Question 13 – Start a new booklet (15 marks) **Marks**

(a) Find the solution of the differential equation $\frac{dy}{dx} = 1 + 4y^2$, given $y(0) = \frac{1}{2}$. **3**

Give your answer in the form $y = f(x)$.

(b) Use the compound angle identities to first simplify and then solve **3**

$\cos 3x - \cos 2x + \cos x = 0$ for $0 \leq \theta \leq \pi$.

(c) A solid is generated when the region bounded by $y = \frac{1}{x}$, $y = 4$, the line $x = 3$ and the y-axis is rotated about the x-axis. **3**

Determine the volume of the solid.

(d) (i) Find $\int \frac{x}{\sqrt{1-x^2}}\,dx$ using the substitution $u = 1 - x^2$ **2**

(ii) Differentiate $x\cos^{-1} x$ with respect to x. **2**

(iii) Hence, find $\int \cos^{-1} x\,dx$ **2**

End of Question 13 – Start a new booklet

Question 14 – Start a new booklet (15 marks) **Marks**

(a) A particle is projected upwards from the top of a building 12 metres high with an initial velocity of 40 ms^{-1}. The particle is launched at an angle of 30 degrees with the horizontal.

Take 9.8 ms^{-2} for the acceleration due to gravity.

(i) If the origin is at the base of the building, show that the displacement vector of the particle is: **3**

$$\underset{\sim}{s}(t) = 20\sqrt{3}\ t\underset{\sim}{i} + (-4.9t^2 + 20t + 12)\underset{\sim}{j}$$

(ii) Show that the time of flight is 4.61 seconds, correct to 2 decimal places. **1**

(iii) Determine the speed, correct to 2 decimal places, and the angle, correct to the nearest degree, at which the particle hits the ground. **3**

(b) Consider a set with n objects without regard to order.

The set is to be split into r groups. The first group is of size n_1, the second is of size n_2, and so forth, with the last group of size n_r.

Note $n_1 + n_2 + \cdots + n_r = n$.

(i) Explain why the number of ways to split the set into r groups is: **1**

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\ldots\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$$

(ii) Show that the expression in part (i) is equal to **2**

$$\frac{n!}{n_1!\,n_2!\ldots n_r!}$$

(iii) A club consisting of four Year 12 students and twelve Year 11 students is randomly divided into groups of four. **3**

Find the probability that each group includes a Year 12 student.

(c) Prove that $\dfrac{\cos 3x}{\cos x} - \dfrac{\sin 3x}{\sin x}$ is independent of x. **2**

End of Paper

Year 12 Mathematics Extension 1 Section I – Answer Sheet

NESA No. | | | | | | | | | |

Name: ______________________________ **Class:** ______________

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

- If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct

A ● B ⊗ C ○ D ○

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1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

2021 Mathematics Extension 1 AP4 Solutions

1) $\text{Proj}_{\underset{\sim}{a}}\underset{\sim}{b} = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{b}|^2}\ \underset{\sim}{b}$

$= \dfrac{(2\times1)+(3\times-4)}{\left(\sqrt{(1)^2+(-4)^2}\right)^2}\times(\underset{\sim}{i}-4\underset{\sim}{j})$

$= -\frac{10}{17}(\underset{\sim}{i}-4\underset{\sim}{j})$ $\boxed{B}$

$= -\frac{10}{17}\underset{\sim}{i}+\frac{40}{17}\underset{\sim}{j}$

2) $\int \sin^2 4x\,dx$

$= \frac{1}{2}\int(1-\cos 8x)\,dx$

$= \frac{1}{2}\left(x-\frac{1}{8}\sin 8x\right)+C$ $\boxed{A}$

$= \frac{x}{2}-\frac{1}{16}\sin 8x + C$

3) $y = 5xe^{2x}$ $\quad\left[\begin{array}{ll} u=5x & v=e^{2x} \\ u'=5 & v'=2e^{2x}\end{array}\right.$ $\boxed{C}$

$\frac{dy}{dx} = 5e^{2x}+10xe^{2x}$

$\frac{d^2y}{dx^2} = 10e^{2x}+10e^{2x}+20xe^{2x}$ $\quad\left[\begin{array}{ll} u=10x & v=e^{2x} \\ u'=10 & v'=2e^{2x}\end{array}\right.$

$= 20e^{2x}+20xe^{2x}$

(C) $\frac{d^2y}{dx^2} - 4y = 20e^{2x}$

LHS $= 20e^{2x}+20xe^{2x}-4(5xe^{2x})$

$= 20e^{2x}+20xe^{2x}-20xe^{2x}$

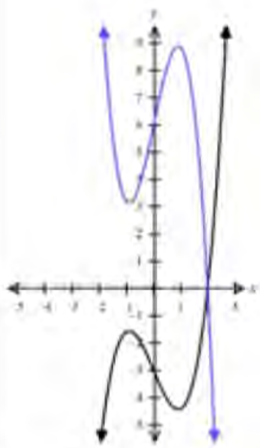
$= 20e^{2x}$

$=$ RHS

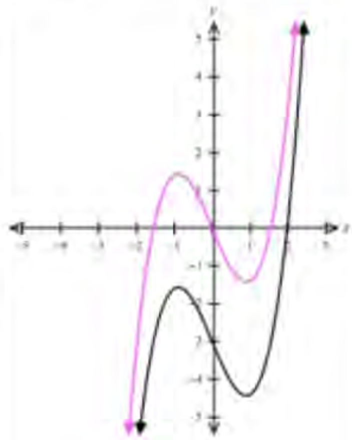
4)

$f(x)$ has 1 root.

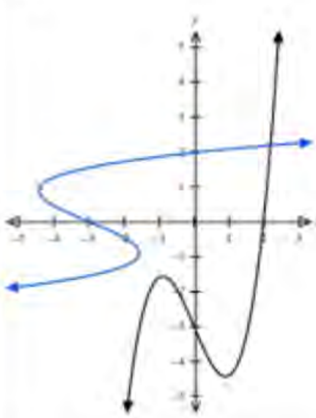
Option A has us doubling the y-values of all points and a reflection in the x-axis which will not increase the number of x-intercepts and hence the roots.



Option B has us moving all points up by 3 units, resulting in 3 roots, which is 2 more than the original function.



Option C switches the x and y coordinates of each point (reflects in line $y=x$) and results in one root, so no more than $f(x)$.



Option D moves each point 3 units to the left which does not change the number of roots.

$\boxed{B}$

5) $\int \tan x \, dx$

$= -\int \frac{-\sin x}{\cos x} \, dx$

$= -\ln|\cos x| + C$

$= \ln|\sec x| + C$

$\boxed{C}$

6) $y = \frac{3+e^{2x}}{5}$ $\therefore$ Inverse : $x = \frac{3+e^{2y}}{5}$

$5x = 3 + e^{2y}$

$5x - 3 = e^{2y}$

$2y = \ln|5x-3|$

$y = \frac{1}{2}\ln|5x-3|$

$\boxed{B}$

7) The x intercept will occur when $y = 0$.

$\cos 2t = 0$

$2t = \frac{\pi}{2}, \frac{3\pi}{2} \ldots .$

$t = \frac{\pi}{4}, \frac{3\pi}{4} \ldots .$

$\boxed{D}$

when $t = \frac{\pi}{4}$ (first x intercept)

$x = -\ln\frac{\pi}{4} \approx 0.24$ (2dp)

8) There are $^{12}C_6$ ways of selecting the 6 directors to go through to the election.

$\boxed{C}$

There are $^{6}P_3$ ways of electing people to the 3 positions

$$^{12}C_6 \times {}^{6}P_3 = \frac{12!}{6!6!} \times \frac{6!}{3!}$$

$$= \frac{12!}{6!\,3!}$$

$$= \frac{^{12}P_6}{3!}$$

9) As $y=2$ makes denominator zero, vertical gradient for points with $y=2$

$\boxed{D}$

As $x=0$ makes numerator zero horizontal gradient for points with $x=0$

This only happens at B and D

By substituting coordinate values into $\frac{dy}{dx}$ option D is correct.

e.g. Test $(3,0)$

$$\frac{dy}{dx} = 3$$

10) $OB \perp CA$

$\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}$ $\qquad$ $\overrightarrow{CA} = \underset{\sim}{a} - \underset{\sim}{c}$

$\boxed{C}$

Since they are perpendicular need to show

$$\overrightarrow{CA} \cdot \overrightarrow{OB} = 0$$

$$(\underset{\sim}{a} - \underset{\sim}{c}) \cdot (\underset{\sim}{a} + \underset{\sim}{c}) = 0$$

Question 11

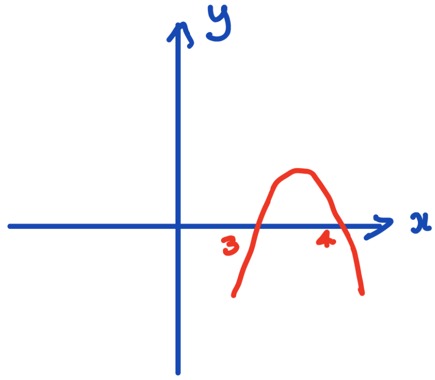
a) $\dfrac{x}{x-3} \leqslant 4 \qquad x \neq 3$

1 mark: shows some understanding

2 marks: Makes significant progress

3 marks: Correct Answer

$$x(x-3) \leqslant 4(x-3)^2$$
$$x(x-3) - 4(x-3)^2 \leqslant 0$$
$$(x-3)\left[x-4(x-3)\right] \leqslant 0$$
$$(x-3)(x-4x+9) \leqslant 0$$
$$(x-3)(12-3x) \leqslant 0$$
$$3(x-3)(4-x) \leqslant 0$$



$\therefore (-\infty, 3) \cup [4, \infty)$

$x < 3$ or $x \geqslant 4 \qquad (x \neq 3)$

b) i) $3\alpha + 3\beta + 3\gamma - 4\alpha\beta\gamma$

$$= 3(\alpha+\beta+\gamma) - 4(\alpha\beta\gamma)$$
$$= 3\left(--\tfrac{4}{2}\right) - 4\left(-\tfrac{5}{2}\right)$$
$$= 6 + 10$$
$$= 16$$

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta\gamma = -\frac{d}{a}$

1 mark - made some progress

2 marks correct soln.

ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

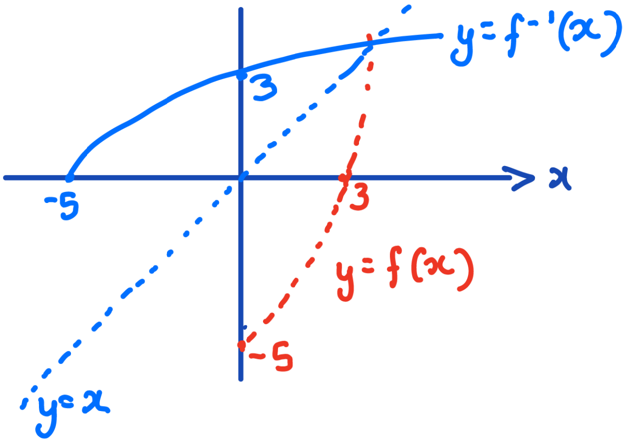
$$= \left(--\tfrac{4}{2}\right)^2 - 2\left(\tfrac{1}{2}\right)$$
$$= 4 - 1$$
$$= 3$$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

1 mark - made some progress

2 marks correct soln.

c) i) Domain Restricted $x \geq 0$

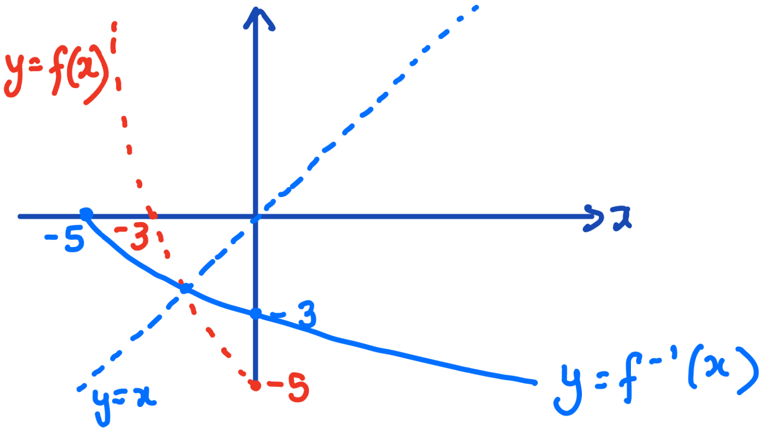


1 mark restricted domain correct

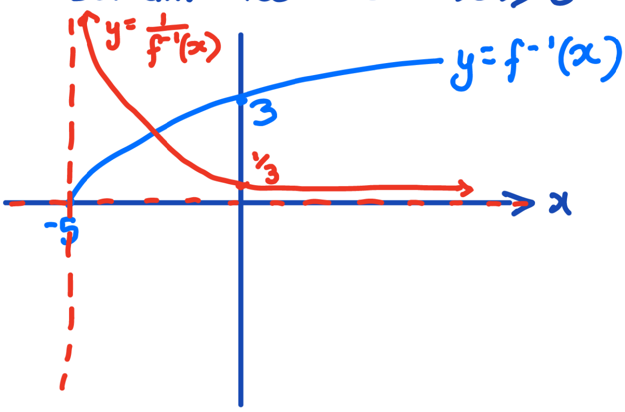
or inverse graph correct with wrong domain

2. All correct

Or Domain Restricted $x \leq 0$



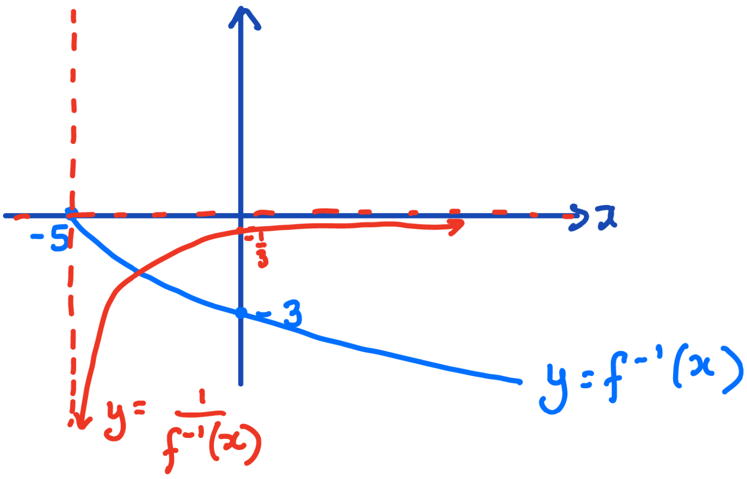
ii) Domain Restricted $x \geq 0$



1 Mark - Asymptotes or y intercept

2 Marks - Correct Soln

Or Domain Restricted $x \leq 0$



d) $P(x) = x^4 - 3x^3 + ax^2 + bx - 1$

$P'(x) = 4x^3 - 9x^2 + 2ax + b$

if divisible by $(x-1)^2$ $\quad P(1) = 0$ and $P'(1) = 0$

i.e. $P(1) = 1 - 3 + a + b - 1 = 0$

$$a + b = 3 \quad \ldots \quad (1)$$

$P'(1) = 4 - 9 + 2a + b = 0$

$$2a + b = 5 \quad \ldots \quad (2)$$

(2) - (1) $\qquad a = 2$

but $a = 2$ into (1) $\qquad b = 1 \qquad \therefore a = 2 \quad b = 1$

1 mark - made some progress

2 marks correct soln.

e) $\underset{\sim}{a} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ $\underset{\sim}{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$= \frac{(3 \times -1) + (-5 \times 4)}{\sqrt{3^2 + (-5)^2} \times \sqrt{(-1)^2 + 4^2}}$$

$$= \frac{-23}{\sqrt{578}}$$

1 mark - made some progress
2 marks correct soln.

$$\theta = 163^\circ$$

Question 12

a i) $4\sin\theta - 3\cos\theta = R\sin(\theta - \alpha)$

$= R(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$

$\therefore R\cos\alpha = 4$

$R\sin\alpha = 3$

1 mark : find R or α

2 marks : correct soln

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{4}$

$\tan\alpha = \frac{3}{4}$

$\alpha = 36°87'$

$(R\cos\alpha)^2 + (R\sin\alpha)^2 = 4^2 + 3^2$

$R^2(\cos^2\alpha + \sin^2\alpha) = 25$

$R^2 = 25$

$R = 5 \quad (R>0)$

$\therefore I = 4\sin\theta - 3\sin\theta = 5\sin(\theta - 36°87')$

ii) Maximum value is 5 (from equation)

It occurs when

$5\sin(\theta - 36.87°) = 5$

$\sin(\theta - 36.87°) = 1$

$\theta - 36.87° = 90°$

$\theta = 126.87°$

1 mark : Find value of θ

2 marks : Justifies Answer

b) $\frac{dV}{dt} = 8\, cm^3 s^{-1}$

$V = \frac{3}{2}(h^2 + 8h)$

$\frac{dV}{dh} = \frac{3}{2}(2h + 8)$

$\frac{dh}{dV} = \frac{2}{3(2h+8)}$

- Using the chain rule

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{2}{3(2h+8)} \times 8$

$= \frac{16}{3(2h+8)}$

1 mark : finds $\frac{dh}{dt}$ or shows some understanding

2 marks : Makes significant progress

3 marks : Correct Answer

- Given $V = 72$

$72 = \frac{3}{2}(h^2 + 8h)$

$h^2 + 8h - 48 = 0$

$(h-4)(h+12) = 0$

$h = 4$ or $h = -12$ (but $h > 0$)

$\therefore h = 4$

$\therefore \frac{dh}{dt} = \frac{16}{3(2 \times 4 + 8)} = \frac{1}{3}\, cm\, s^{-1}$

$\therefore$ The rate of change of the depth of the water is $\frac{1}{3}$ cm/s when $V = 72\, cm^3$.

c) $u = \cos 2\theta$

$$\frac{du}{d\theta} = -2\sin 2\theta$$

$$du = -2\sin 2\theta \, d\theta$$

$$\sin 2\theta \, d\theta = -\frac{1}{2} \, du$$

when $\theta = \frac{\pi}{2}$ $\quad u = \cos \pi = -1$

when $\theta = \frac{3\pi}{4}$ $\quad u = \cos \frac{3\pi}{2} = 0$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \cos^2 2\theta \sin 2\theta \, d\theta = \int_{-1}^{0} -\frac{1}{2} u^2 \, du$$

$$= -\frac{1}{2} \int_{-1}^{0} u^2 \, du$$

$$= -\frac{1}{2} \left[\frac{u^3}{3} \right]_{-1}^{0}$$

$$= -\frac{1}{2} \left(0 - -\frac{1}{3} \right)$$

$$= -\frac{1}{6}$$

1 mark : shows some understanding

2 marks : Makes significant progress

3 marks : Correct Answer

d) Prove true for $n=1$

$$\text{LHS} = \frac{1}{2} \qquad \text{RHS} = 2 - \frac{1+2}{2^1}$$

$$= 2 - \frac{3}{2}$$

$$= \frac{1}{2}$$

LHS = RHS $\therefore$ true for $n=1$

Assume true for $n=k$

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{k}{2^k} = 2 \quad \frac{k+2}{2^k}$$

Need to prove true for $n=k+1$

i.e. Need to prove

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} = 2 - \frac{k+1+2}{2^{k+1}}$$

$$= 2 - \frac{k+3}{2^{k+1}}$$

$$\text{LHS} = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}}$$

$$= 2 + \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}} \quad \text{(by assumption)}$$

$$= 2 - \frac{k+2}{2^k} \times \frac{2}{2} + \frac{k+1}{2^{k+1}}$$

$$= 2 - \frac{2k+4}{2^{k+1}} + \frac{k+1}{2^{k+1}}$$

$$= 2 + \frac{-k-3}{2^{k+1}}$$

1 mark : Show true for $n=1$

2 marks : Significant progress

3 marks : Correct soln

$$= 2 - \frac{k+3}{2^{k+1}}$$

$$= RHS$$

Hence, by the principle of mathematical induction, the claim is true for all $n \geq 1$

e) Considering the pairs of numbers which have a difference of 3:

5, 8   6, 9   7, 10

If we selected 5, 6, 7 then we would not have any pairs with a difference of 3. But on our next selection, either 8, 9 or 10 we will have one of the pairs above.

By the time you have selected 4 digits, you must have at least one pair.

1 mark - made some progress
2 marks correct soln.

Question 13

a) $\frac{dy}{dx} = 1 + 4y^2$

Separating Variables

$$\frac{1}{1+4y^2}\,dy = dx$$

Integrating both sides

$$\int \frac{1}{1+(2y)^2}\,dy = \int dx$$

$$\frac{1}{2}\tan^{-1} 2y = x + C$$

Sub $y(0) = \frac{1}{2}$

$$\frac{1}{2}\tan^{-1} 2\left(\frac{1}{2}\right) = 0 + C$$

$$C = \frac{1}{2}\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{8}$$

$$\therefore \frac{1}{2}\tan^{-1} 2y = x + \frac{\pi}{8}$$

$$\tan^{-1} 2y = 2x + \frac{\pi}{4}$$

$$2y = \tan\left(2x + \frac{\pi}{4}\right)$$

$$y = \frac{1}{2}\tan\left(2x + \frac{\pi}{4}\right)$$

1 mark : shows some understanding

2 marks : Makes significant progress

3 marks : Correct Answer

b) Using $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

$$\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)$$

$$= \frac{1}{2}\left[\cos\left(\frac{3x+x}{2} + \frac{3x-x}{2}\right) + \cos\left(\frac{3x+x}{2} - \frac{3x-x}{2}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{6x}{2}\right) + \cos\left(\frac{2x}{2}\right)\right]$$

$$= \frac{1}{2}\left[\cos 3x + \cos x\right]$$

$$\therefore \cos 3x + \cos x = 2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)$$

$$\cos 3x - \cos 2x + \cos x = 0$$

$$(\cos 3x + \cos x) - \cos 2x = 0$$

$$2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) - \cos 2x = 0$$

$$2\cos 2x \cos x - \cos 2x = 0$$

$$\cos 2x\,(2\cos x - 1) = 0$$

$\therefore \cos 2x = 0$ or $2\cos x - 1 = 0$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\cos x = \frac{1}{2}$

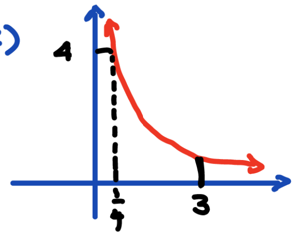
$x = \frac{\pi}{4}, \frac{3\pi}{4}$ $x = \frac{\pi}{3}$

1 mark : shows some understanding

$$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{3}$$

2 marks : Makes significant progress

3 marks : Correct Answer

c) 

The solid is made up of a cylinder of radius 4 and height $\frac{1}{4}$ and a solid formed when the area under the curve $y = \frac{1}{x}$ $\frac{1}{4} \leq x \leq 3$ is rotated.

$$V = \left(\pi 4^2 \times \frac{1}{4}\right) + \pi \int_{\frac{1}{4}}^{3} \frac{1}{x^2}\, dx$$

$$= 4\pi + \pi \int_{\frac{1}{4}}^{3} x^{-2}\, dx$$

$$= 4\pi + \pi \left[\frac{x^{-1}}{-1}\right]_{\frac{1}{4}}^{3}$$

$$= 4\pi + \pi \left[-\frac{1}{x}\right]_{\frac{1}{4}}^{3}$$

$$= 4\pi + \pi \left(-\frac{1}{3} - -4\right)$$

$$= 4\pi + \frac{11\pi}{3}$$

$$= \frac{23\pi}{4} \text{ units}^3$$

1 mark : shows some understanding

2 marks : Makes significant progress

3 marks : Correct Answer

d i)

Let $u = 1 - x^2$

$\frac{du}{dx} = -2x$

$-\frac{1}{2} du = x\, dx$

$$\int \frac{x}{\sqrt{1-x^2}}\, dx = -\frac{1}{2} \int \frac{1}{\sqrt{u}}\, du$$

$$= -\frac{1}{2} \int u^{-\frac{1}{2}}\, dy$$

$$= -\frac{1}{2}\left(2u^{\frac{1}{2}}\right) + C$$

$$= -\sqrt{u} + C$$

$$= -\sqrt{1-x^2} + C$$

1 mark : makes some progress

2 marks : correct solution

ii) $u = x$ $\quad v = \cos^{-1}x$

$u' = 1$ $\quad v' = -\dfrac{1}{\sqrt{1-x^2}}$

1 mark : makes some progress

2 marks : correct solution

$$\frac{d}{dx}\left(x\cos^{-1}x\right) = \cos^{-1}x - \frac{x}{\sqrt{1-x^2}}$$

iii) $$\int \cos^{-1}x - \frac{x}{\sqrt{1-x^2}}\,dx = x\cos^{-1}x$$

$$\int \cos^{-1}x\,dx = x\cos^{-1}x + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

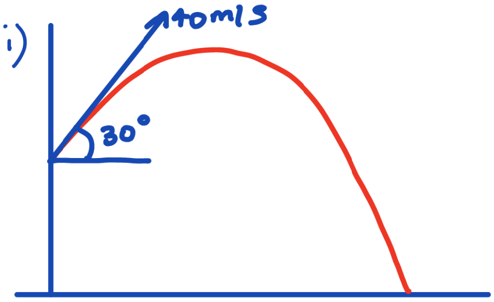
$$\int \cos^{-1}x\,dx = x\cos^{-1}x - \sqrt{1-x^2} + C$$

1 mark : makes some progress

2 marks : correct solution

Question 14

a i)



when $t = 0$

$$\underset{\sim}{v}(t) = 40\cos 30\,\underset{\sim}{i} + 40\sin 30\,\underset{\sim}{j}$$

$$= 40 \times \frac{\sqrt{3}}{2}\,\underset{\sim}{i} + 40 \times \frac{1}{2}\,\underset{\sim}{j}$$

$$= 20\sqrt{3}\,\underset{\sim}{i} + 20\,\underset{\sim}{j}$$

1 mark : shows some understanding

2 marks : Makes significant progress

3 marks : Correct Answer

$$a(t) = -9.8\,\underset{\sim}{j}$$

$$v(t) = \int -9.8\,\underset{\sim}{j}\;dt$$

$$= -9.8t\,\underset{\sim}{j}$$

when $t = 0$

$$20\sqrt{3}\,i + 20\,\underset{\sim}{j} = c$$

$$\therefore\; v(t) = -9.8t\,\underset{\sim}{j} + 20\sqrt{3}\,\underset{\sim}{i} + 20\,\underset{\sim}{j}$$

$$= 20\sqrt{3}\,\underset{\sim}{i} + (-9.8t + 20)\,\underset{\sim}{j}$$

$$s(t) = \int 20\sqrt{3}\,\underset{\sim}{i} + (-9.8t + 20)\,\underset{\sim}{j}\;dt$$

$$= 20\sqrt{3}t\,\underset{\sim}{i} + \left(-\frac{9.8t^2}{2} + 20t\right)\underset{\sim}{j} + c$$

when $t = 0$ $\quad s(t) = 12\,\underset{\sim}{j}$

$$12\,\underset{\sim}{j} = c$$

$$\therefore\; s(t) = 20\sqrt{3}t\,\underset{\sim}{i} + (-4.9t^2 + 20t)\,\underset{\sim}{j} + 12\,\underset{\sim}{j}$$

$$s(t) = 20\sqrt{3}t\,\underset{\sim}{i} + (-4.9t^2 + 20t + 12)\,\underset{\sim}{j}$$

ii) The time of flight is when the y component of the displacement is 0.

i.e. $-4.9t^2 + 20t + 12 = 0$

$$t = \frac{-20 \pm \sqrt{20^2 - 4(4.9)12}}{2(-4.9)}$$

$$= -0.53 \text{ or } 4.61 \qquad (\text{But } t > 0)$$

$\therefore \quad t = 4.61$ seconds.

iii) $v(t) = 20\sqrt{3}\,\underset{\sim}{i} + (-9.8t + 20)\underset{\sim}{j}$

when $t = 4.61$

$$v(4.61) = 20\sqrt{3}\,\underset{\sim}{i} + (-9.8 \times 4.61 + 20)\underset{\sim}{j}$$

$$= 20\sqrt{3}\,\underset{\sim}{i} + -25.178\,\underset{\sim}{j}$$

$\therefore$ Velocity $= \sqrt{(20\sqrt{3})^2 + (-25.178)^2}$

$= 42.83$ m/s

Angle of Inclination: $\tan\theta = \dfrac{-25.178}{20\sqrt{3}}$

$\theta = -36^\circ$

$\therefore$ The particle hits the ground with a velocity of 42.83 m/s at an angle of 144° with the horizontal

1 mark: shows some understanding

2 marks: Makes significant progress

3 marks: Correct Answer

b i) With n objects, first combine n_1 of them with $\binom{n}{n_1}$ possibilities.

Left with $n-n_1$ objects and combine n_2 of them, with $\binom{n-n_1}{n_2}$.

The pattern continues until there are $n-n_1-n_2-\dots n_{r-1}$ objects and combine the remaining n_r of them with $\binom{n-n_1-n_2\dots n_{r-1}}{n_r}$ possibilities.

∴ The number of ways to split the objects into r groups is the product of the possible number of each of the r subsets.

ii) $$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\dots n_{r-1}}{n_r}$$

$$= \frac{n!}{n_1!\,\cancel{(n-n_1)!}} \times \frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-n_1-n_2)!}} \times \frac{\cancel{(n-n_1-n_2)!}}{n_3!\,\cancel{(n-n_1-n_2-n_3)!}} \times\dots\times \frac{\cancel{(n-n_1-n_2-\dots-n_{r-1})!}}{n_r!\,(n-n_1-n_2\dots-n_{r-1}-n_r)!}$$

$$= \frac{n!}{n_1!\,n_2!\dots n_r!}$$

Since $n_1+n_2+\dots+n_r=n$

$(n-n_1-n_2-\dots-n_{r-1}-n_r)! = 0! = 1$

1 mark - shows some understanding

2 marks - correct solution

iii) Applying the result in (ii), the number of possible groupings $= \dfrac{16!}{4!\,4!\,4!\,4!}$

Number of possible groupings where each of the four groups contains a Year 12 student :

Each of the Year 12 students must be assigned to one of the four groups.

That leaves 12 students to be put into four groups of 3 (each of which can be ordered in 4! ways)

$$4! \times \frac{12!}{3!3!3!3!}$$

$$\text{Probability} = \frac{4! \times \frac{12!}{3!3!3!3!}}{\frac{16!}{4!4!4!4!}}$$

$$= \frac{64}{455}$$

1 mark : shows some understanding

2 marks : Makes significant progress

3 marks : Correct Answer

c) $\frac{\cos 3x}{\cos x} - \frac{\sin 3x}{\sin x}$

$= \frac{\sin 3x \sin x - \sin 3x \cos x}{\cos x \sin x}$

$= - \frac{\sin (3x - x)}{\frac{1}{2} \times 2 \sin x \cos x}$

$= \frac{- \sin 2x}{\frac{1}{2} \times \sin 2x}$

$= -2$ (which is independent of x)

1 mark - shows some understanding

2 marks - correct solution